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COMBINING DATA FOR IMPROVED WEIBULL PARAMETER ESTIMATION.

by

10 S.B. Batdorf ~~and~~ George Sines

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Weibull parameters are often used by experimentalists to characterize the fracture behavior of structural materials. Frequently several different types of tests are performed, and theory indicates that the parameters for each type should be the same, or have some specific relations to each other. In such cases, it is advantageous to combine the data to find the parameters best representing all the data. The present paper discusses techniques for carrying out this objective and analyzes two examples in detail to → next page

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COMBINING DATA FOR IMPROVED WEIBULL PARAMETER ESTIMATION

by

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ABSTRACT

Weibull parameters are often used by experimentalists to characterize the fracture behavior of structural materials. Frequently several different types of tests are performed, and theory indicates that the parameters for each type should be the same, or have some specific relation to each other. In such cases, it is advantageous to combine the data to find the parameters best representing all the data. The present paper discusses techniques for carrying out this objective and analyzes two examples in detail to illustrate the principles involved. The first example compares uniaxial and equibiaxial fracture statistics. The second example discusses fracture statistics for single fibers of several different lengths.

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INTRODUCTION

According to weakest link theory, the probability of failure P_f of a specimen of volume V uniformly loaded in simple tension σ must take the form

$$P_f(\sigma) = 1 - \exp[-V\, f(\sigma)] \qquad (1)$$

The functional form of $f(\sigma)$ depends on the particular weakest link theory under consideration. Weibull (Ref. 1,2) not only derived Eq. (1), but pointed out that $f(\sigma)$ has a simple physical significance; it is the number of flaws per unit volume having a strength equal to or less than σ. For this reason, he used the symbol $n(\sigma)$. In addition, he introduced a simple parametric representation of $n(\sigma)$,

$$n(\sigma) = \left(\frac{\sigma-\sigma_u}{\sigma_o}\right) = k(\sigma-\sigma_u)^m \qquad (2)$$

$$n(\sigma) = \left(\frac{\sigma}{\sigma_o}\right)^m = k\,\sigma^m \qquad (3)$$

Eq. (2) is a special case of Eq. (3) in which σ_u is taken to be 0. These equations have been widely used, and experimentalists often present their results in terms of the Weibull parameters representing the best fit, rather than giving the original data.

If the probability of survival is given by

$$P_s(\sigma) = 1-P_f(\sigma) = \exp[-V\ k(\sigma-\sigma_u)^m] \tag{4}$$

it follows that

$$\ell n\ \ell n\ P_s^{-1}(\sigma) = \ell n\ Vk + m\ \ell n(\sigma-\sigma_u) \tag{5}$$

The usual way of determining the Weibull parameters is to plot

$$y = \ell n\ \ell n\ P_s^{-1} \tag{6a}$$

against

$$x = \ell n(\sigma-\sigma_u) \tag{6b}$$

According to Eq. (5), the result is a straight line of slope m and y-intercept equal to ℓn Vk. Thus a least squares fit of a straight line to the experimental points placed on a Weibull plot furnishes an estimate of the Weibull parameters. The probability of survival P_s is usually taken to be (Ref. 1)

$$P_s = \frac{N-j+1}{N+1} \tag{7}$$

where N is the number of test points and j is the ordinal number of a particular specimen ranked in order of increasing strength.

Not infrequently, an experimental program involves more than one kind of test. For example, several different sizes or shapes of specimens may be involved, or the test series may include several different stress states. As a result of experimental scatter, the Weibull parameters found for the various subsets of test data can be expected to differ somewhat, even though as material properties they should be the same. Because more test data give more accurate results, it seems desirable to combine the test data whenever feasible to get the best possible values. The purpose of the present note is to call attention to the merits of such a procedure, and to suggest techniques by which it may be accomplished. First some basic theory will be outlined, after which a couple of illustrative examples will be presented. The second example involves *extensions* to the theory.

We digress for a moment to acknowledge that there exist in the literature more sophisticated techniques for Weibull parameter estimation than that described above (see for instance Ref. 3-6). Such techniques probably give more accurate results and often include desirable treatments of confidence intervals. It is quite likely that some of these techniques could be expanded to include the analysis of combined data. The present authors chose to generalize the conventional approach described above because they believe that the resulting technique is simple enough that the majority of engineers could be persuaded to employ it.

Let us assume we have several sets of data α, β, γ...consisting of pairs of data points $(x_{\alpha_1}, y_{\alpha_1})$, $(x_{\alpha_2}, y_{\alpha_2})$ etc..., which should plot on straight lines. The equation for such a straight line is

$$y = m\,x + b \tag{8}$$

where m is the slope and b is the y intercept. Let us further assume that it is known from theory that all the slopes are the same, but that the intercepts b_α, b_β, etc...are in general different. We seek a least squares fit to all the data subject to above constraints.

Let us define a sum of squares for each data set as follows:

$$I_\nu\left(m, b_\nu\right) = \sum_i \left(y_{\nu i} - mx_{\nu i} - b_\nu\right)^2 \tag{9}$$

and

$$I\left(m, b_\alpha, b_\beta ...\right) - \sum_\nu I_\nu\left(m, b_\nu\right) \tag{10}$$

The quantity I is the sum of squares of all the differences between the observed values of y and the values predicted by straight line representations of the various subsets of data α, β, etc. We minimize the sum of squares by solving the simultaneous equations:

$$\frac{1}{2}\frac{\partial I}{\partial m} = \sum_i \sum_\nu x_{\nu i}\left(y_{\nu i} - mx_{\nu i} - b_\nu\right) = 0 \tag{11}$$

$$\frac{1}{2}\frac{\partial I}{\partial b_\nu} = \sum_i \left(y_{\nu i} - mx_{\nu i} - b_\nu\right) = 0 \qquad (\nu = \alpha, \beta ...) \tag{12}$$

Solving this set of simultaneous equations with the aid of Cramer's rule, we obtain for m

$$m = \frac{\sum_\nu \sum_i x_{\nu i}\, y_{\nu i} - \sum_\nu \left(\sum_i x_{\nu i}\right)\left(\sum_i y_{\nu i}\right)/n_\nu}{\sum_\nu \sum_i x_{\nu i}{}^2 - \sum_\nu \left(\sum_i x_{\nu i}\right)^2/n_\nu} \tag{13}$$

where n_ν represents the number of data points of type ν. The b's can be obtained directly from Eq. (12):

$$b_\nu = \frac{1}{n_\nu}\left(\sum_i y_{\nu i} - m \sum_i x_{\nu i}\right) = \overline{y}_\nu - m\,\overline{x}_\nu \tag{14}$$

Here $\overline{x}_\nu$ and $\overline{y}_\nu$ are the mean values of the coordinates for each data type; i.e., they are the coordinates of the centroid of all type ν data points. The summations appearing in Eq. (13) and (14) are automatically computed in statistical subroutines incorporated into many hand calculators as well as large digital computers when data pairs $x_{\nu i}$ and $y_{\nu i}$ are entered into the appropriate registers.

Eqs. (13) and (14) are easily reconstructed and used without referring back to the present paper (although perhaps we shouldn't tell you this) by remembering the following points:

(1) The slope m found in a conventional linear regression analysis is given by

$$m = \frac{\Sigma xy - \Sigma x\, \Sigma y/n}{\Sigma x^2 - (\Sigma x)^2/n} = \frac{p}{q} \tag{15}$$

What we have just shown is that in case the data are to be fitted to several lines, the common slope is given by

$$m = \frac{p_\alpha + p_\beta + \ldots}{q_\alpha + q_\beta + \ldots} \tag{16}$$

which is a very logical generalization of Eq. (15).

(2) Equation (14) tells us that to get the desired least squares fit, all we have to do is to pass lines of this slope through the centroids of the data subsets.

Example 1 - Biaxial Tension

According to Weibull theory, if fracture data in simple tension obey the two-parameter Weibull equation

$$P_S(\sigma) = \exp[-V k\sigma^m] \tag{17}$$

the fracture statistics for polyaxial tension obeys the equation

$$P_S(\sigma_1,\sigma_2,\sigma_3) = \exp\left[-V F\left(m, \frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_1}\right) k \sigma_1{}^m\right] \tag{18}$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses, and F is a correction factor which depends upon the stress ratios and m. A closed form expression for F is given in Ref. 7.

We note in passing that in going from uniaxial to polyaxial stress states, the utility of the Weibull parameters is only preserved if we use Weibull's two-parameter representation, Eq. (17). The reasons for this are discussed in Ref. 8. However, in many cases, the use of 0 as the value of σ_u does not entail much loss in accuracy, especially when not many data points are available.

Weibull's theory for polyaxial stress states is based on the implicit assumption that only the component of stress normal to the crack plane contributes to its fracture (Ref. 9). Recently, it has been shown (Ref. 10) that fracture statistics based on more realistic fracture criteria also imply the Eq. (17-18) relationship, but with a factor F which depends upon the fracture criterion employed. Since the proper fracture criterion for mixed loading conditions is not settled yet,we treat F as an unknown parameter.

In an experiment designed to evalute F for equal or biaxial tension (Ref.11) 10 bars of high strength alumina were fractured in 4-point bending and ten discs were fractured employing concentric ring loading. The volumes and areas in the test sections of the two types of specimens were the same and the correction factor relating uniform tension and pure bending are the same for the uniaxial and equibiaxial case (Ref.12). Thus the maximum stress values at fracture can be used to obtain the best values of Weibull's parameters and the factor F.

The test dat are given in Table 1. The same data are displayed as a Weibull plot in Fig. 1, where the coordinates are defined by Eq. (6a,b) with the assumption that $\sigma_u = 0$. The values of m as obtained from linear regression analyses of the two sets of data were 14.06 for the uniaxial case and 14.54 for the equibiaxial case. According to theory,

the two lines should be parallel and separated vertically by a distance equal to ℓn F. Note that finding a single value for m is not only desirable for greater accuracy, but essential to the determination of F.

Employing the relations shown in Eq. (6), Eqs. (13) and (14) were applied to the data in Table 1 with the following results:

$$m = 14.296$$

$$b_\alpha = -53.585$$

$$b_\beta = -52.376$$

The vertical separation of the two lines is

$$b_\beta - b_\alpha = 1.209 = \ell n\ F \tag{19}$$

whence

$$F = 3.35$$

Reference 7 shows that according to Weibull's theory, F should be approximately 6.6. Inclusion of shear in the fracture criterion leads to lower values of F and improved agreement with experiment (Ref. 6).

To convert b_α into a Weibull parameter, we note that if the probability of survival in simple tension is given by Eq. (17), the probability of survival of a rectangular bar in pure bending is (Ref. 1,9)

$$P_s = \exp\left[-\frac{V\ k}{2(m+1)}\sigma^m\right] \tag{20}$$

We note also that the y-intercept is the value of $\ell n\, \ell n\, P_s^{-1}$ when $x = \ell n\, \sigma = 0$ or $\sigma = 1$. *Thus*

$$b_\alpha = \ell n \left[\frac{V\, k}{2(m+1)} \right] \tag{21a}$$

or

$$k = \frac{2(m+1)}{V} \exp[b_\alpha] \tag{21b}$$

Using the previously determined values of m and b_α, and $V = 0.442$ in^3, we obtain $k = 3.70 \times 10^{-22}$ as the value of the second Weibull parameter when the volume and stress are expressed in cubic inches and ksi respectively.

Example 2 - Single Fiber Testing

A number of experiments have been carried out to determine the statistics of fracture of single fibers and the dependence of the statistics on fiber length. According to Weibull's two parameter representation,

$$P_s(\sigma) = \exp[-L\, k\, \sigma^m] \tag{22}$$

From the preceding discussion, we would expect that Weibull plots of such data would result in different straight lines for different lengths, but that the line for L_β could be obtained from the line for L_α simply by a vertical shift equal to $\ell n\ (L_\beta/L_\alpha)$. The actual behavior usually differs somewhat from this due to real world considerations.

First of all, the stressed length of fiber is necessarily larger than the nominal test section. In the case of high modulus fibers, the stress in the fibers extends somewhat into the region of the test fixture due to shear lag effects. Consequently, unless precautions are taken to exclude all test data corresponding to failure outside the test section, an effective length Δ must be added to each fiber length (Ref. 13,14). This results in a vertical separation equal to $\ell n[(L_\beta+\Delta)/(L_\alpha+\Delta)]$. In addition, at a sufficiently large tensile stress, the fiber begins to tear loose from the support structure, greatly complicating the stress analysis and the interpretation of the fracture data.

One of the most recent and extensive tests on single fiber strength was conducted by Barr, Chwastiak and Didchenko (Ref. 15). These authors have kindly made the original data available to the present writers. An examination of the data for their high strength carbon fibers suggests that the process of tearing away from the support

structure began about σ = 2.7 GPa. Accordingly, the analysis undertaken in the present paper is limited to stress levels below this. The fracture data for length L = 1.4 mm, 3.2 mm, 10 mm, 20 mm, and 40 mm are contained in Table 2. In the cases of 3.2, 20 and 40 mm, the data were too numerous to permit easy plotting so they were reduced by including only every other failure.

Applying Eqs. (13) and (14) to these data, we obtained the results in Table 3. In this Table are listed the various sums appearing in Eq. (13), the m values that would be obtained for each length considered individually, and the consolidated m value given by Eq. (13). Also shown are the intercepts obtained using Eq. (14). Fig. 2 displays on a Weibull plot the experimental data, the centroids, and the best straight lines constrained to have a common slope. The dotted lines through the centroids indicate the results obtained when a linear regression analysis is made for each length considered individually.

As might be expected, the vertical separation of the lines is less than that given by simple theory, indicating the presence of an effective length Δ (Table 4). This must be taken into account as we turn to the task of finding the value of Weibull's second parameter, k.

To do this, we need to develop some additional theory. Since the value $b_{\alpha 0}$ just obtained for the α data y-intercept has not been required to satisfy the theoretical separation, we look for a value

$$b_\alpha = b_{\alpha 0} + s \tag{23}$$

which does satisfy this relation. Then, all the intercepts can be expressed simply:

$$b_\nu = b_{\alpha 0} + s + \delta_\nu \tag{24}$$

where

$$\delta_\nu = \ell n[(L_\nu + \Delta)/(L_\alpha + \Delta)] \tag{25}$$

Thus

$$y_\nu = m\, x_\nu + b_\nu = m\, x_\nu + b_{\alpha 0} + \delta_\nu + s \tag{26}$$

Equation (26) is the most general expression for straight-line representations of the Weibull plots of the data which satisfy the vertical separation requirement. So we now seek to minimize the mean square discrepancy between the data and the theory. That is, we desire to minimize $J(s,\Delta)$ by proper choice of s and Δ, where

$$J(s,\Delta) = \sum_\nu \sum_i \left(y_{\nu i} - m\, x_{\nu i} - b_\nu\right)^2 \tag{27}$$

Minimization of J with respect of s can be accomplished analytically by setting $\partial J/\partial s = 0$. First we rewrite Eq. (27) in the form

$$J = \sum_{\nu} \sum_{i} \left[\left(y_{\nu i} - m\, x_{\nu i} - b_{\nu 0}\right) + \left(b_{\nu} - b_{\nu 0}\right)\right]^2 \tag{28}$$

When the binomial between the rectangular brackets in Eq. (28) is squared, we obtain three terms. The first term does not contain s, and is simply equal to the I of Eq. (10). The cross-term vanishes by virtue of Eq. (12). As a result

$$\frac{\partial J}{\partial s} = \frac{\partial K}{\partial s} = 0 \tag{29}$$

where

$$K = \sum_{\nu}\left(b_{\nu} - b_{\nu 0}\right)^2 \tag{30}$$

Carrying out the indicated operation and making use of Eq. (24), we obtain

$$\Sigma\, n_{\nu}\left(b_{\nu} - b_{\nu 0}\right) = \sum_{\nu} n_{\nu}\left(b_{\alpha 0} + \delta_{\nu} - b_{\nu 0} + s\right) = 0 \tag{31}$$

The value of s satisfying this equation is $s = s^*$ where

$$s^* = -b_{\alpha 0} + \frac{\sum_{\nu} n_{\nu}\left(b_{\nu 0} - \delta_{\nu}\right)}{\sum_{\nu} n_{\nu}} \tag{32}$$

and the minimum value of K with respect to s is

$$K = \sum_{\nu} n_{\nu}\left(b_{\alpha 0} + \delta_{\nu} - b_{\nu 0} + s^*\right) \tag{33}$$

Since δ_ν is a function of the effective length Δ, s^* and therefore K are also functions of Δ. If we have some independent means of knowing what Δ should be, this value should be inserted, together with s^*, into Eq. (24) to determine b_ν. If not, Δ can be inferred from the data by minimizing K with respect to Δ. This can be done graphically by calculating K for a series of values of Δ, and taking that value of Δ for which K is a minimum. The latter procedure was employed in the present paper.

Several values were selected for Δ in the neighborhood of 1 mm. For each value of Δ, δ_ν was calculated using Eq. (25), s^* was obtained using Eq. (32), and K was evaluated using Eq. (33). In Fig. 3, K is plotted against Δ. The minimum occurs when $\Delta = \overline{\Delta} = 1.02$ mm. The corresponding value of s^* is 0.0105. The y-intercepts b_ν were then found using Eq. (24). The results are listed in Table 1.5, and shown in Fig. 2 as inverted triangles on the y axis.

The Weibull parameter k can be obtained using any one of the data sets as follows. From Eq.

$$b_\nu = \ell n[k(L_\nu + \overline{\Delta})] \tag{34a}$$

or

$$k = \exp(b_\nu)/(L_\nu + \overline{\Delta}) = \exp(b_{\alpha 0} + s^*)/(L_\alpha + \overline{\Delta}) \tag{34b}$$

The above equation yields $k = 3.08 \times 10^{-3}$ when lengths are expressed in millimeters and stresses are in GPa.

DISCUSSION

Combining data sets to obtain improved estimates of the Weibull parameters involves a combination of standard (though possibly not widely known) statistical procedures and weakest link theory. The latter is to some degree in a state of flux and sometimes more than one approach is possible. Successful application of the technique therefore involves tailoring the approach to the specific situation with due consideration for applicable theory. For example, in Example 1, the vertical separation between the two lines in a Weibull plot is not uniquely specified by theory, so we determined it by experiment. On the other hand, in Example 2, theory does give the vertical separations, subject however to an uncertainty resulting from the influence of an unknown effective length Δ, which has to be determined experimentally. In Example 2, we employed Weibull's two-parameter representation because there was no evidence of curvature in the Weibull plots. If there had been, it would have been advantageous to employ his three-parameter representation. Choosing σ_u to minimize the least square discrepancy between theory and experiment could be accomplished graphically, as was done in the determination of Δ.

In the case of Example 1, the value found for m based on the use of all the data was essentially the mean of the values of m determined for uniaxial and biaxial data treated separately. Since the number of test points was the same for both cases, one might be led to suspect that m is an average of m_α, m_β, etc. weighted in some manner to reflect the numbers of data points, n_α, n_β, etc. That such is not the case can be demonstrated as follows.

It will be recalled that the slope of the ν data in a Weibull plot is given by

$$m_\nu = p_\nu / q_\nu \tag{15}$$

when only those data are considered, whereas when all the data are considered the common slope is

$$m = \Sigma\, p_\nu / \Sigma\, q_\nu \tag{16}$$

Dividing both the numerator and denominator of the latter equation by $\Pi\, q_\nu$ we obtain

$$m = \left[\frac{m_\alpha}{q_\beta q_\delta \ldots} + \frac{m_\beta}{q_\alpha q_\delta \ldots} + \ldots \right] Q \tag{35}$$

where

$$Q = \frac{\Pi\, q_\nu}{\Sigma\, q_\nu} \tag{36}$$

Thus the ratio of the weighting factor for m_ν to that for m_α is q_α/q_ν. Since from (15)

$$q = \Sigma xy - \Sigma x \, \Sigma y/n$$

it is clear that the weighting factors are not determined solely by the numbers n_ν; other statistical properties of the data are also involved.

Whether an unfamiliar procedure will be adopted depends largely on the cost/benefit tradeoff, so a few words on this subject are in order. In the case of Gaussian distributions, the fractional uncertainties in mean and standard deviation are approximately equal to $N^{-0.5}$ where N is the number of observations. Weibull distributions can be regarded as skewed Gaussian distributions, so the same rule is roughly applicable. This means that greater accuracy should result from pooling the data. In the case of Example 1, the slopes found for the individual sets of data differed by only about 3% which is far less than the uncertainty in the final result, and made it necessary for the purposes of our illustration to use more significant figures than are warranted by the data. It could be argued that a different example would be preferable. However, the authors chose in both examples to use real data readily available to them, since the sole purpose was to demonstrate a least squares approach to combining data.

Also it should be noted that, as Robinson has pointed out (Ref. 6) when $\sigma_u = 0$, there is a simpler method of pooling the data. This is to reduce the stresses in different data sets to a common basis, as for instance, by normalizing to the average or the median stress, and then using least squares to determine m. This is equivalent to using a single point fit for determining the intercept b_ν instead of a least squares fit. Obviously, this is less precise than the least squares approach for obtaining both m and b_ν described here, and will not in general lead to the same result. However, in many cases, the differences will be far less than the statistical uncertainty $N^{-0.5}$ mentioned earlier.

With regard to how much work is involved in processing data in the manner suggested here, this depends of course on both the hardware and software employed. The present authors used a card programmable hand-held computer with a memory capacity sufficient to store all fracture data and a built-in statistical subroutine giving the sums appearing in Eq. (15). With such equipment, the time needed to obtain the results used in Example 1 is of the order of 10 minutes and for Example 2 an hour or two. The time for thinking through the program the first time must be added to this, something on the order of an hour should suffice for programming. The authors feel that, in most cases with which they are familiar, the improvement in the results of the data reduction obtained following procedures such as those recommended herein greatly outweigh the costs in additional effort.

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TABLE 1

Plate Failure Stresses (ksi)

j	Uniaxial	Biaxial
1	36.696	34.728
2	38.212	34.874
3	38.401	35.153
4	40.203	37.103
5	41.060	37.673
6	41.498	37.879
7	41.595	38.072
8	43.371	38.826
9	43.632	41.320
10	46.196	41.815

TABLE 2

Fiber Failure Stresses (GPa)

	L(mm)	1.4(α)	3.2(β)	10(γ)	20(δ)	40(ε)
j	N	39	57	33	30	19
1		1.67	0.965	1.24	0.651	0.887
2		1.91	1.19	1.72	0.891	1.01
3		2.53	1.66	1.75	1.14	1.22
4		2.69	1.74	1.82	1.29	1.31
5			2.16	1.84	1.47	1.40
6			2.32	1.91	1.63	1.51
7			2.36	1.94	1.76	1.57
8			2.39	1.96	1.83	1.91
9			2.45	2.19	1.90	1.95
10			2.47	2.23	1.91	2.12
11			2.65	2.46	2.05	2.20
12				2.49	2.09	2.35
13				2.54	2.22	2.40
14				2.54	2.30	2.56
15				2.59	2.36	2.62
16				2.63	2.47	
17				2.69	2.61	

TABLE 3

Fiber Calculations

Data Subset	α	β	γ	δ	ε
Σy	-11.4484	-26.5591	-23.7647	-21.8706	-12.9607
Σy^2	33.9114	70.1282	45.7751	41.0337	23.8137
n	4	11	17	17	15
Σx	3.0777	7.1931	12.6754	8.9811	8.0464
Σx^2	2.5225	5.7603	10.1454	7.0459	5.9723
Σxy	-8.4025	-14.9389	-14.8523	-6.1364	-2.4306
m_ν	2.6289	2.2986	4.1287	2.3543	2.7308
m(consolidated) = 2.6681					
$b_{\nu 0}$	-4.9149	-4.1592	-3.3873	-2.6961	-2.2953

TABLE 4

Vertical Separation

ν	α	β	γ	δ	ε
Experimental ($b_{\nu 0} - b_{\alpha 0}$)	0	0.7557	1.5276	2.2188	2.6196
Theoretical ($\ln \frac{L_\nu}{L_\alpha}$)	0	0.8267	1.9661	2.6593	3.3524

TABLE 5

Optimum y-intercepts

ν	α	β	γ	δ	ε
b_ν	-4.9044	-4.3483	-3.3885	-2.7427	-2.0741

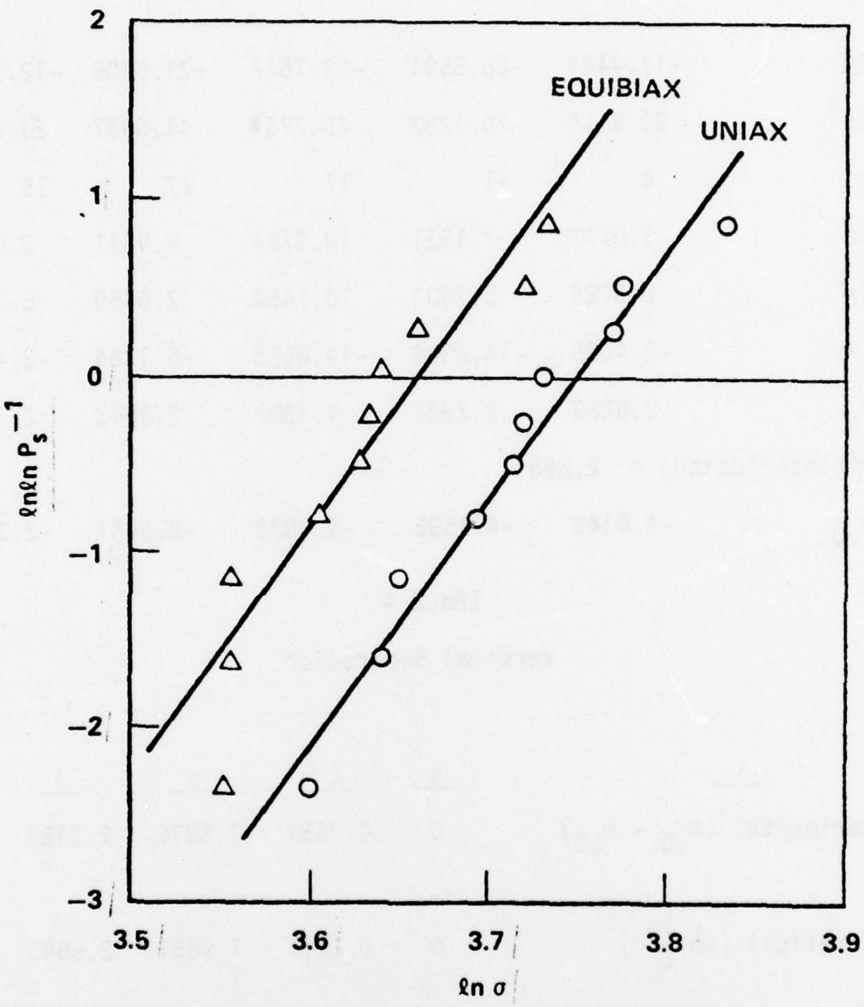

FIGURE 1 Weibull Plots of Fracture Data for Uniaxial and Equibiaxial Stress States.

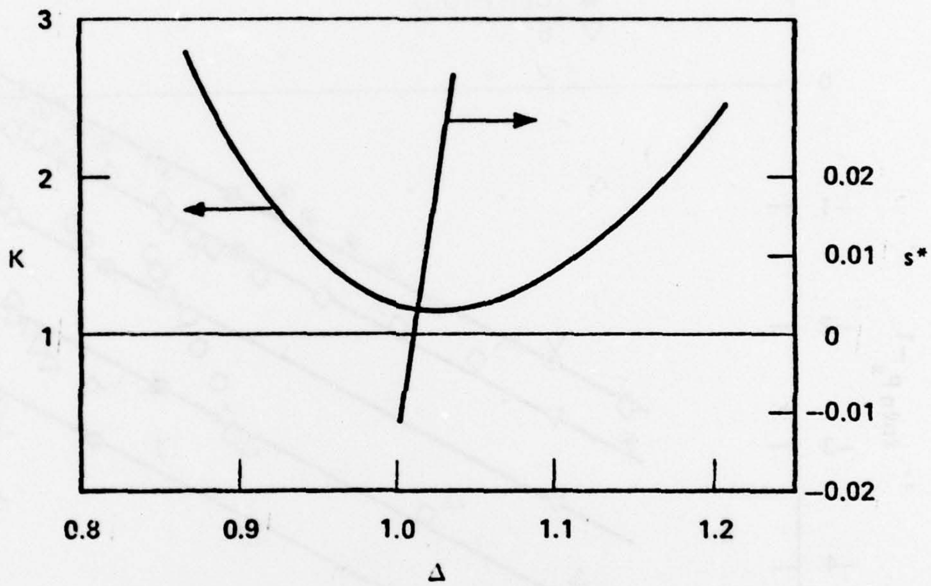

FIGURE 2 Graphical Determination of K and s^*.

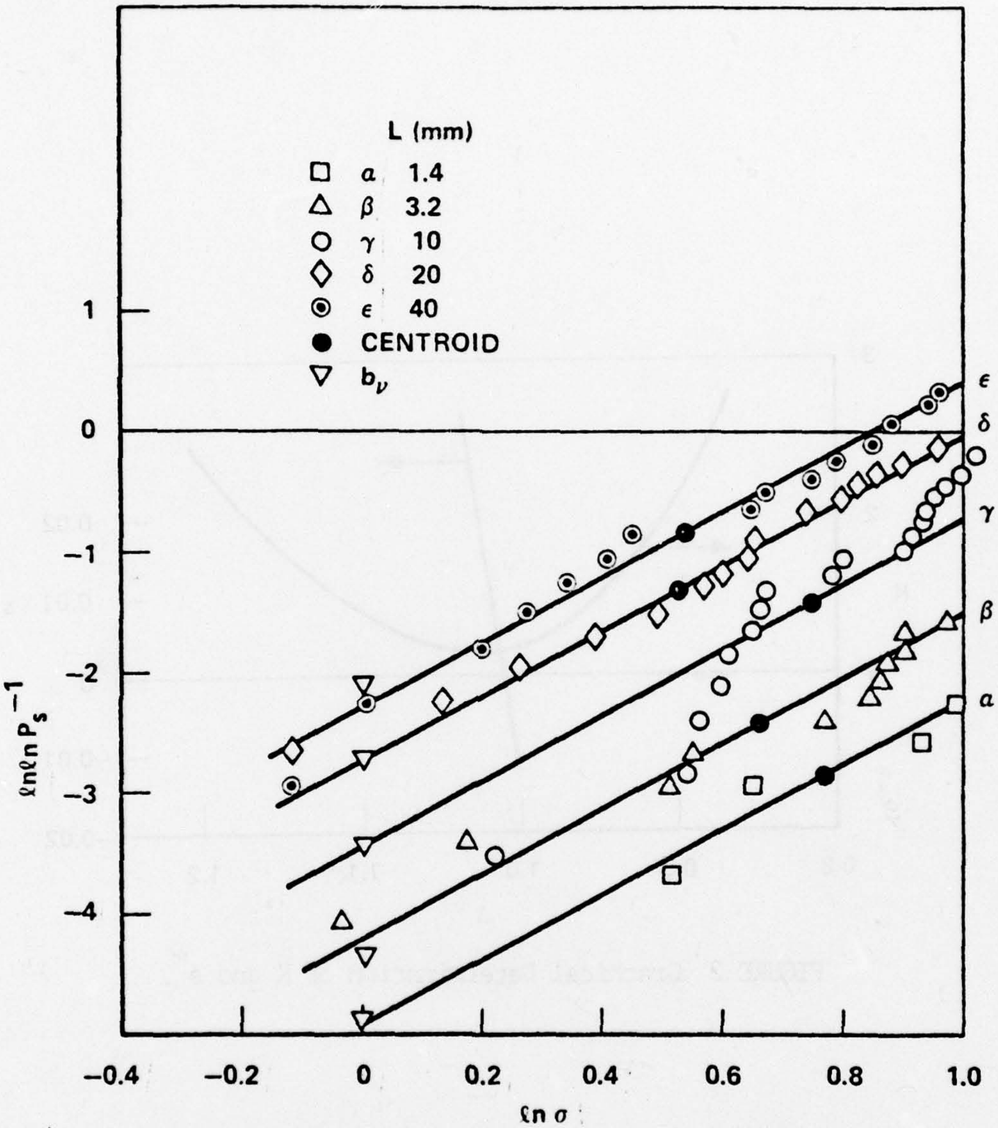

FIGURE 3 Weibull Plots of 5 Carbon Fiber Lengths.